Markov Process

$$P(X_{n+1}=j \mid X_n=a_n, X_{n-1}=a_{n-1}, \dots X_0=a_0)$$

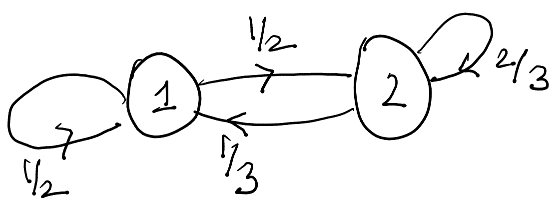
$$= P(X_{n+1}=j \mid X_n=a_n)$$

I will write this as $p_n(i,j)$ the one-step transition prob. at time n.

If $p_n(i,j) = p(i,j)$ $\forall n$ then the Markov Process is called TIME HOMOGENEOUS

Sometimes I will also write p_{ij}

Two state MC:



$p_{11} = 1/2$ $\quad p_{12} = 1/2$

$p_{21} = 1/3$ $\quad p_{22} = 2/3$

Will collect all of these things in a matrix

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

N-step transition probabilities

$$P(X_0=a_0, X_1=a_1, X_2=a_2, \dots, X_n=a_n)$$

will write this in terms of one-step transition probabilities.

$f(a_0, a_1, \dots, a_{n-1}, a_n) = g(a_{n-1}, a_n)$!

$$= P(X_n = a_n \mid X_0 = a_0, X_1 = a_1, \dots, X_{n-1} = a_{n-1})$$
$$P(X_0 = a_0, X_1 = a_1, \dots X_{n-1} = a_{n-1})$$

$$= p_{n-1}(a_{n-1}, a_n)$$
$$P(X_0 = a_0, X_1 = a_1, \dots X_{n-1} = a_{n-1})$$

$$\dots = p_{n-1}(a_{n-1}, a_n) \dots p_0(a_0, a_1)$$

Ex

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.6 & 0.3 & 0.1 \\ 1 & 0.3 & 0.3 & 0.4 \\ 2 & 0.4 & 0.1 & 0.5 \end{array}$$

What does the transitions graph look like?

$$P(X_0 = 1, X_1 = 0, X_2 = 2 \mid X_0 = 1)$$

$$= P(X_2 = 2, X_1 = 0 \mid X_0 = 1) = \frac{P(X_2 = 2, X_1 = 0, X_0 = 1)}{P(X_0 = 1)} =$$

"peel it like an onion"

Recall markov process $\{X_n\}$, $X_n \in S$ where

the STATE space.

Then Thm: We can compute the quantities $P(X_0 = a_0, \dots X_n = a_n)$ using just the transition probabilities $p_i(a_i, a_{i+1})$:

$$P(X_0 = a_0, \dots\dots X_n = a_n) = P(X_0 = a_0) \prod_{i=0}^{n-1} p_i(a_i, a_{i+1})$$

Pf:

$$P(X_0 = a_0, \dots\dots X_n = a_n)$$

$$= P(X_n = a_n \mid X_{n-1} = a_{n-1}, \dots \quad X_0 = a_0)$$
$$P(X_{n-1} = a_{n-1}, \dots\dots \quad X_0 = a_0)$$

$$= p_{n-1}(a_{n-1}, a_n) \dots\dots \quad P(X_1 = a_1 \mid X_0 = a_0)$$
$$P(X_0 = a_0)$$

$$= \prod_{i=0}^{n-1} p_i(a_i, a_{i+1}) P(X_0 = a_0)$$

Let's assume $p_n(x, y) = p(x, y)$

(Time homogeneity)

and assume $S = \{1, \dots, N\}$

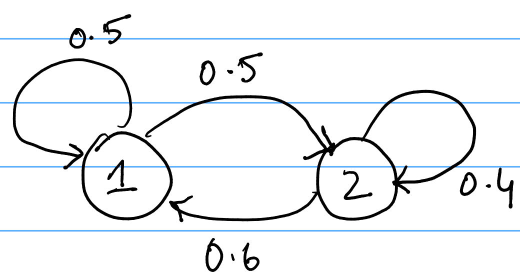
$P_{N\times N} = [\,p(i,j)\,]$ Matrix.

Then <u>Thm</u>:

$$P(X_{n+m} = j \mid X_m = i) = [P^n]_{ij}$$

where P^n is the n^{th} power of P.

Ex:



$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}$$

Find $P(X_2 = 2 \mid X_0 = 1)$

To find $[P^2]_{1,2}$

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 3/5 & 2/5 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} + \frac{3}{10} & \frac{1}{4} + \frac{1}{5} \\ \frac{3}{10} + \frac{6}{25} & \frac{3}{10} + \frac{4}{25} \end{bmatrix} = \begin{bmatrix} \frac{22}{40} & \frac{9}{20} \\ \frac{15+12}{50} & \frac{15+8}{50} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{22}{40} & \frac{18}{40} \\ \frac{27}{50} & \frac{23}{50} \end{bmatrix}$$

So $\left[P^2\right]_{1,2} = \frac{18}{40}$

We will refer to this prob as $p^{(n)}(i,j)$ $\quad n = 0, 1, 2, \cdots$

where $p^{(0)}(i,j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Ex: Suppose $P(X_0 = 1) = P(X_0 = 2) = \frac{1}{2}$

Then find $P(X_2 = 2)$.

$$= P(X_2 = 2 \mid X_0 = 1)\, P(X_0 = 1) + P(X_2 = 2 \mid X_0 = 2)\, P(X_0 = 2)$$

$$= \frac{18}{40}\,\frac{1}{2} + \frac{23}{50}\,\frac{1}{2}$$

$$P^n_{ij} = \sum_{j=1}^{N} P^{n-1}_{ik} P_{kj}$$

$$= \sum_j P(X_{n-1} = k \mid X_0 = i)\, P(X_1 = j \mid X_0 = k)$$

Time homo.

$$= \sum_k P(X_{n-1} = k \mid X_0 = i)\, P(X_n = j \mid X_{n-1} = k)$$

$$= \sum_{j} \frac{P(X_{n-1}=k, X_0=i)}{P(X_0=i)} \overset{\downarrow \text{Markov prop.}}{P(X_n=j \mid X_{n-1}=k, X_0=i)}$$

$$= \sum_{j} \frac{P(\cancel{X_{n-1}=k}, X_0=i)}{P(X_0=i)} \frac{P(X_n=j, X_{n-1}=k, X_0=i)}{P(\cancel{X_{n-1}=k}, X_0=i)}$$

$$= \frac{P(X_n=j, X_0=i)}{P(X_0=i)} = P(X_n=j \mid X_0=i)$$

Used Law of total probability here.

Notice:

1) $\sum_j P_{ij} = 1$ (You must end up in some state j, Law of total prob.

2) $\sum_j P_{ij}^n = 1$ (Same reason)

Such a matrix with row sums $= 1$ is called

A STOCHASTIC MATRIX

HW questions.

3.4.1 3.4.2 3.1.1 3.1.2

3.2.1 3.2.2

3.3 has the following sections:

1) Inventory Model
2) Ehrenfest Urn model (Maxwell's demon)
Probably a bit too advanced for a math class.
3) Markov Chain in Genetics
4) Fischer - Wright model in genetics.
5) A discrete queuing markov chain. - not that interesting.

3.3 Examples

Inventory model.

Stock replenished if current stock $< L$ (low level)

Replenishment level H.

$- H$

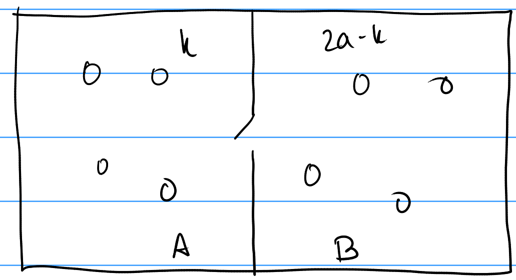
$x_n \to$ Stock amount right before replenishment

$- L$

$$X_{n+1} = \begin{cases} X_n - \xi_n & L \le X_n \le H \\ H - \xi_n & X_n < H \end{cases}$$

amount of demand.

Ehrenfest Urn Model



Let Y_n = # of balls in urn A

$X_n = Y_n - a$ "fluctuation about a"

$$P(X_n = k \mid X_{n-1} = j) = \begin{cases} \frac{a-j}{2a} & k = j+1 \\ \frac{a+j}{2a} & k = j-1 \end{cases}$$

Does $\lim_{n \to \infty} \frac{X_n}{2a}$ exist? We asked a similar question about Polya's urn.

Genetics Markov chains

. . . . N individuals

k Type A alleles

$N-k$ Type B

Fisher's model: at each time step, resample the allele with prob. proportional to # of alleles.

X_n = # of type A alleles at time n.

$$X_n = \sum_{i=1}^{n} \xi_i$$

where $P(\xi_i = 1) = \dfrac{X_{n-1}}{N}$

NOTE: 0 and N are absorbing states.

(One of the 1st instances of absorbing states)

- Can talk about mutation probabilities. $A \xrightarrow{\alpha} B$ $\quad B \xrightarrow{\beta} A$

$$P(\xi_i = 1) = \frac{X_{n-1}}{N}(1-\alpha) + \frac{N - X_{n-1}}{N}\beta$$

- Can also talk about selection pressure. (more likely to choose a type A mate)

3.4 First step analysis for absorption

$$P = \begin{vmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{vmatrix}$$

$$\begin{tikzcd} 0 & 1 \arrow[l, "\alpha"'] \arrow[r, "\gamma"] \arrow[loop below, "\beta"] & 2 \end{tikzcd}$$

Rem: I sometimes do gamblers ruin and show how the presence of absorbing states changes things.

Its clear that the absorbing states. $\{0\}, \{2\}$

$T = \inf\{ n : X_T \in \{0,2\} \}$ TIME TO ABSORPTION

$u = P(X_T = 0 \mid X_0 = 1)$

$= P($ absorbed in 0 rather than 2 $\mid X_0 = 1)$

$v = E[T \mid X_0 = 1]$ = "average TIME to absorption"

Lets use conditional expectation

$$E[1_{\{X_T = 0\}} \mid X_0] = E\left[E[1_{X_T=0} \mid X_0, X_1] \,\middle|\, X_0 \right]$$

At $X_0 = 1$ the LHS $= u$

$$\sum_{k=0}^{2} \mathbb{E}\left[\mathbb{1}_{X_T=0} \mid X_0=1, X_1=k\right] \mathbb{P}(X_1=1 \mid X_0=1)$$

$$= 1\,\mathbb{P}(X_1=0 \mid X_0=1) + \beta\,\mathbb{E}\left[\mathbb{1}_{\{X_T=0} \mid X_0=1, X_1=1\right]$$

$$+\ 0\,\mathbb{P}(X_1=0 \mid X_0=1)$$

$$= \alpha + \beta\,\mathbb{E}\left[\mathbb{1}_{\{X_T=0\}} \mid X_0=1, X_1=1\right] = \alpha + \beta u$$ (but maybe this doesn't convince you)

$$= \alpha + \beta \sum_{k=2}^{\infty} 1\ \mathbb{P}(X_k=0, X_{k-1}=1, \ldots X_2=1 \mid X_0=0, X_1=1)$$

$$= \alpha + \beta \sum_{k=2}^{\infty} \mathbb{P}(X_k=0, \ldots X_2=1 \mid X_1=1)$$

(Using Markov Property here)

$$= \alpha + \beta \sum_{k=1}^{\infty} \mathbb{P}(X_{k-1}=0, \ldots, X_1=1 \mid X_0=1)$$

$$= \alpha + \beta u$$

$$\Rightarrow u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha+\gamma}$$ since $\alpha+\beta+\gamma=1$

★ ask on HW or quiz. (the following)

Show that

$$\vartheta = \mathbb{E}[T \mid X_0 = 1] = \mathbb{E}\Big[\mathbb{E}[T \mid X_0 = 1, X_1] \,\Big|\, X_0 = 1\Big]$$

$$= \sum_{k=0}^{2} \mathbb{E}[T \mid X_0 = 1, X_1 = k] \; \mathbb{P}(X_1 = k \mid X_0 = 1)$$

$$= 1(\alpha + \gamma) + \beta\left(\mathbb{E}[T \mid X_0 = 1, X_1 = 1]\right)$$

show $= 1 + E[T \mid X_0 = 1]$ using an ∞ series.

$$= \alpha + \gamma + \beta(1 + \vartheta)$$

$$\Rightarrow \vartheta = \frac{1}{1-\beta} \qquad \vartheta = E[X] \quad X \sim \mathrm{Geom}(1-\beta)$$

Suppose we have a 4 state chain

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \min\{n \mid X_n = 0 \text{ or } X_3 = 3\}$$

$$u_i = P(X_T = 0 \mid X_0 = i) \qquad i = 1, 2$$

$\{1,2\}$ Transient class. $\{0\}, \{3\}$ absorbing classes

$$u_i = E\left[T \mid X_0 = i\right] \qquad i = 1,2$$

$$u_1 = P_{10} + P_{11} u_1 + P_{12} u_2 + 0\, P_{13}$$

$$u_2 = P_{21} u_1 + P_{22} u_2 + 0 \cdot \cancelto{0}{P\, u_3} + 1 P_{20}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}}_{Q} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix}}_{R}$$

Similarly

$$\vartheta_1 = 1(P_{10} + P_{13}) + (1+\vartheta_1) P_{11} + (1+\vartheta_2) P_{12}$$

$$\vartheta_2 = 1(P_{23} + P_{20}) + (1+\vartheta_1) P_{21} + (1+\vartheta_2) P_{22}$$

$$\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} P_{10} + P_{13} \\ P_{20} + P_{23} \end{bmatrix} + \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}}_{Q} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

$$(1 - Q)\vec{u} = \begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix} \qquad \vec{u} = (1-Q)^{-1} \begin{bmatrix} P_{10} \\ P_{20} \end{bmatrix}$$

$$(1 - Q)\vec{\vartheta} = (1 - Q^{-1}) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It's worth doing an example here:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find $P(X_T = 0 \mid X_0 = 2)$

$$\vec{u} = \left[I - \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 0.7 & -0.2 \\ -0.3 & 0.7 \end{bmatrix}^{-1} \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$$

$$= \frac{1}{(0.7^2 - (0.2)(0.3))} \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$$

$$= \frac{100}{43} \begin{bmatrix} \frac{30}{100} \\ \frac{19}{100} \end{bmatrix}$$

Classification of Finite chains

Draw pictures for

1) Irreducible chains: It is possible to get from i to j in some # of steps

2) Communication classes
  - Transient
  - Absorbing

In general arrange $\vec{P}$ as follows:

$$\vec{P} = \begin{array}{r} \text{transient} \rightarrow \\ \text{absorbing} \rightarrow \end{array} \left[\begin{array}{c:c} Q & R \\ \hdashline \vec{0} & I \end{array}\right]$$

$X_n \in \{0, \ldots, N\}$

Transient $\in \{0, 1, \ldots, r-1\}$

Absorbing $\in \{r, \ldots, N\}$

For finite state MCs that are irreducible and have at least one absorbing state

Transient $= (\text{Absorbing})^c$

In general

Transient $:= \{ i \in \{0, \ldots, N\} \mid P_{ij}^{(n)} \to 0$ as $n \to \infty \}$

Here: TRAN $=$ S $\setminus$ ABS.

Ok, back to P, Q, R, I

Q is $r \times r$

Q is $N-r+1 \times r$

R is $r \times N-r+1$

I is $N-r+1 \times N-r+1$

In the 4 state MC.

$$P = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 0 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 0 \\ 3 \end{matrix} & \begin{bmatrix} P_{11} & P_{12} & P_{10} & P_{13} \\ P_{21} & P_{22} & P_{20} & P_{23} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Define $U_{ik} = \mathbb{P}(X_T = k \mid X_0 = i)$

$$U_{ik} = \sum_j \mathbb{P}(X_T = k \mid X_0 = i, X_1 = j)\, \mathbb{P}(X_1 = j \mid X_0 = i)$$

$$= \sum_{\substack{j \in ABS \\ j \neq k}} P_{ij} \cdot 0 + P_{ik} + \sum_{j \in TRANS} P_{ij} U_{jk}$$

$$U_{ik} = P_{ik} + \sum_{j=0}^{r-1} P_{ij} U_{jk}$$

s a matrix equation

$$\begin{bmatrix} U_{1k} \\ \vdots \\ U_{r-1k} \end{bmatrix} = \begin{bmatrix} R_{1k} \\ \vdots \\ R_{r-1k} \end{bmatrix} + \begin{bmatrix} Q_{11} & \cdots & Q_{1r-1} \\ & & \\ & & Q_{r-1r-1} \end{bmatrix} \begin{bmatrix} U_{1k} \\ \vdots \\ U_{r-1k} \end{bmatrix}$$

$$\Rightarrow \quad (I-Q)\begin{bmatrix} U_{1k} \\ \vdots \\ U_{v-1k} \end{bmatrix} = \begin{bmatrix} R_{1k} \\ \vdots \\ R_{v-1k} \end{bmatrix}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} U_{1k} \\ \vdots \\ U_{v-1k} \end{bmatrix}}_{U_k} = (I-Q)^{-1} \begin{bmatrix} R_{1k} \\ \vdots \\ R_{v-1k} \end{bmatrix}$$

Then as a matrix equation,

$$U_k = (I-Q)^{-1} R \,,$$

where $U_{ik} = \mathbb{P}(X_T = k \mid X_0 = i)$

Similarly

$$v_i = \mathbb{E}[T \mid X_0 = i]$$

$$= \sum_{j=0}^{v-1} (1 + v_j) P_{ij} + \sum_{j \in ABS} P_{ij}$$

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad V = Q \cdot V + \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\vec{1}}$$

$$V = (I-Q)^{-1} \cdot \vec{1}$$

$$V = (I-Q)^{-1}(V+R)$$

☆ The rat model is a fun HW to do.

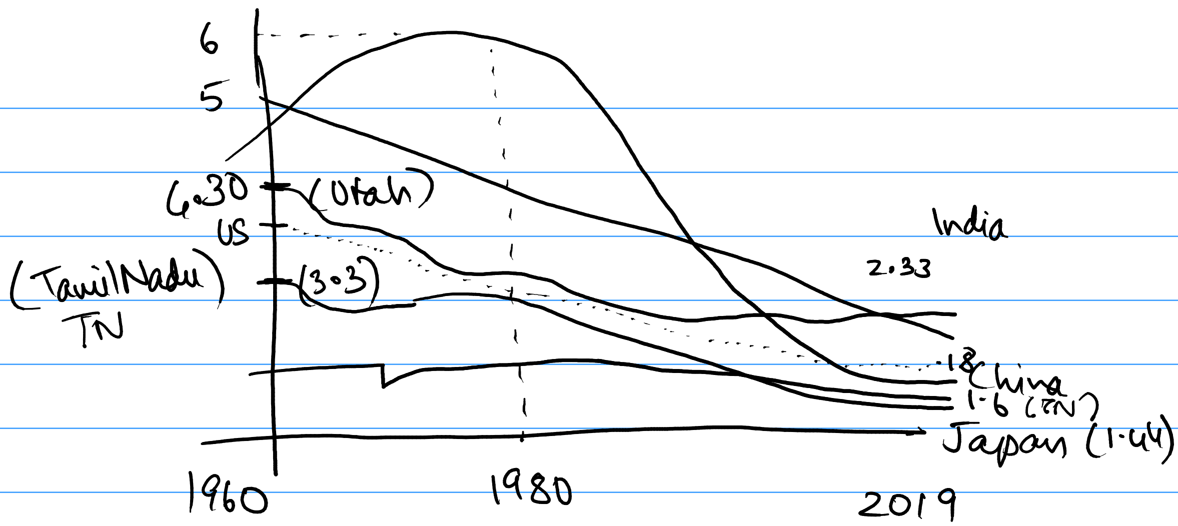
<u>Modeling</u>: How to model population growth

One way: assume all women have a constant birth rate over life

$: \dfrac{\text{Total \# of babies}}{\text{Total \# of women}}$.

Better model: each woman has a "state"

E_0 : Prepuberty
E_1 : Single
E_2 : Married
E_3 : Divorced
E_4 : Widowed
E_5 : Not reproducing or dead.



Your textbook does not use the word "fertility" and instead uses fecundity.

$$P = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4, E_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 0 & 0.9 & 0 & 0 & 0 & 0.1 \\ 0 & 0.5 & 0.4 & 0 & 0 & 0.1 \\ 0 & 0.4 & 0.5 & 0 & 0 & 0.1 \\ 0 & 0 & 0.4 & 0 & 0.5 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

"You can estimate this matrix from demographic data!"

What duration of her life does a woman spend in her child rearing age?

Recall women in 5 states

E_0 (prepuberty) $\quad$ E_3 Divorced

E_1 Single $\quad$ E_4 Widowed

E_2 Married $\quad$ E_5 Dead

How long does a woman spend in her childrearing stage in her lifetime?

$$P = \begin{array}{c|cccccc} & E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \\ \hline E_0 & 0 & 0.9 & 0 & 0 & 0 & 0.1 \\ E_1 & 0 & 0.5 & 0.4 & 0 & 0 & 0.1 \\ E_2 & 0 & 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ E_3 & 0 & 0 & 0.4 & 0.5 & 0 & 0.1 \\ E_4 & 0 & 0 & 0.4 & 0 & 0.5 & 0.1 \\ E_5 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{array}$$

T_2 = Amount of time spent in state 2 $= \sum_{i=0}^{\infty} 1_{\{X_i = 2\}}$

$W_i = \mathbb{E}[T_2 \mid X_0 = i]$

Clearly $w_0 =' \mathbb{E}\left[T_2 \mid X_1 = E_1\right] \mathbb{P}(X_1 = E_1 \mid X_0 = E_0)$

$\qquad + \ \mathbb{E}\left[T_2 \mid X_1 = E_5\right] \mathbb{P}(X_2 = E_5 \mid X_0 = E_0)$

$w_0 = 0.9 w_1 + 0.1 w_5$

(of course w_5 will be 0 eventually since we can never exit w_5) Set $w_5 = 0$

$$
\begin{array}{c|cccccc}
 & E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \\
\hline
E_0 & 0 & 0.9 & 0 & 0 & 0 & 0.1 \\
E_1 & 0 & 0.5 & 0.4 & 0 & 0 & 0.1 \\
E_2 & 0 & 0 & 0.6 & 0.2 & 0.1 & 0.1 \\
E_3 & 0 & 0 & 0.4 & 0.5 & 0 & 0.1 \\
E_4 & 0 & 0 & 0.4 & 0 & 0.5 & 0.1 \\
E_5 & 0 & 0 & 0 & 0 & 0 & 1.0
\end{array}
$$

$w_1 = 0.5 w_1 + 0.4 w_2 + 0.1 \cancel{w_5}^{\,0}$

$w_2 = 1 + 0.6 w_2 + 0.2 w_3 + 0.1 w_4$

$w_3 = 0.4 + 0.5 w_3$

$w_4 = 0.4 w_2 + 0.5 w_4$

absorbing

Already $P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$ absorbing

If $\vec{w} = \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ Then we have

$\vec{w} = Q \cdot \vec{w} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ comes from the fact you're counting time only in state E_2

$\vec{w} = (I - Q)^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

So we have seen that a lot of problems can be solved this way.

$$\vec{U}_k = (I-Q)^{-1} \begin{bmatrix} P_{1k} \\ P_{2k} \\ \vdots \\ P_{n-r\,k} \end{bmatrix} \qquad U_{ik} = P(X_T = k \mid X_0 = i)$$

$$\vec{V} = (I-Q)^{-1} \vec{1} \qquad \vec{V}_i = E[T \mid X_0 = i]$$

$$\vec{W}_k = (I - Q^{-1}) \vec{\delta}_k \qquad (\vec{\delta}_k)_i = \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & \text{otherwise} \end{cases}$$

3.5 Some special Markov Chains

3.5.1 2 state Markov Chain.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 2/5 & 3/5 \\ 1 & 4/5 & 1/5 \end{array} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$0<a<1$
$0<b<1$

lets find the eigenvalues of P

characteristic polynomial

$$\det(\lambda I - P) = 0 \Rightarrow (\lambda - 1 + a)(\lambda - (1-b)) - ab = 0$$

Set $\lambda - 1 = \hat{\lambda}$

$$\Rightarrow \hat{\lambda}^2 + \hat{\lambda}(a+b) + ab - ab = 0$$

$$\Rightarrow \hat{\lambda}(\hat{\lambda} + (a+b)) = 0$$

$\lambda=1$ will always be an eval.

$$\Rightarrow \hat{\lambda} = 0 \text{ or } -a-b \Rightarrow \boxed{\lambda = 1 \text{ OR } 1-a-b}$$

$1-a-b<1$

Eigenvectors.

$\vec{x}A = \lambda\vec{x}$

$$\underbrace{[u \ v]}_{\text{left eigenvector}} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \overset{\lambda=1}{1}\,[u \ v]$$

$$= [u \ v] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [u \ v] \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -au + bv = 0 \Rightarrow v = \frac{au}{b}$$

ONLY 1 equation.

u & v represent probab.

Thus choose u and v st $\underbrace{u+v}=1$

$\Rightarrow \quad u + \frac{au}{b} = 1 \quad \Rightarrow \quad u = \frac{b}{a+b}$

and $v = \frac{a}{a+b}$

$\lambda = 1$ has evec. $\left[\frac{b}{a+b} \quad \frac{a}{a+b}\right]$

Similarly consider $\lambda = 1-a-b$

$$(u,v)\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} = (u,v)\underbrace{(1-a-b)}_{\text{2nd eigenvalue}}$$

$$\Rightarrow (u,v)\begin{pmatrix} b & a \\ b & a \end{pmatrix} = (0,0)$$

$\Rightarrow \quad ub + vb = 0 \quad$ So $u = -v \quad$ Might as well choose $u = \frac{1}{a+b}$

So $\lambda = 1-a-b$ has $\left(\frac{1}{a+b}, \frac{-1}{a+b}\right)$ as the eigenvector.

Set $Q = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{1}{a+b} & \frac{-1}{a+b} \end{pmatrix} \begin{array}{l} \to \lambda = 1 \\ \to \lambda = 1-a-b \end{array} = \frac{1}{a+b}\underbrace{\begin{pmatrix} b & a \\ 1 & -1 \end{pmatrix}}_{\tilde{Q}}$

Q is the matrix of row eigenvectors

$$\hat{\tilde{Q}}^{-1} = \frac{1}{(-b-a)}\begin{pmatrix} -1 & -a \\ -1 & b \end{pmatrix} = \frac{1}{a+b}\begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix}$$

$$\hat{Q}^{-1}\hat{\tilde{Q}} = I \Rightarrow \tilde{Q}^{-1}(a+b)\underbrace{\frac{1}{a+b}\tilde{Q}}_{Q} = I$$

$$\Rightarrow Q^{-1} = (a+b)\,\hat{\tilde{Q}}^{-1} = \begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix}$$

standard inverse calculation

Then
$$QPQ^{-1} = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{1}{a+b} & \frac{-1}{a+b} \end{pmatrix}\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} \frac{1}{a+b}(b-\cancel{ab}+\cancel{ab}) & \frac{a}{a+b} \\ (1-a-b)\frac{1}{a+b} & \frac{(1-a-b)(-1)}{a+b} \end{pmatrix}\begin{pmatrix} 1 & a \\ 1 & -b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a(1-a-b)+b(1-a-b)}{a+b} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1-a-b \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \Lambda$$

Or in other words $\underbrace{P}_{\text{transition matrix}} = Q^{-1}\Lambda Q$

HW: Verify that

$$\underbrace{P^n}_{\text{n step transition matrix}} = \frac{1}{a+b}\overbrace{\begin{bmatrix} b & a \\ b & a \end{bmatrix}}^{A} + \frac{(1-a-b)^n}{a+b}\underbrace{\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}}_{B}$$

$$P^n = \underbrace{\frac{1}{a+b}\Big[A}_{\text{only this will remain}} + \underbrace{(1-a-b)}_{<1 \text{ in abs. value}}{}^n B \Big]$$

Since $0 < a < 1$, $0 < b < 1$ $\Rightarrow$ $0 < a+b < 2$

$\Rightarrow 1 > 1-a-b > -1$

$\Rightarrow$ $(1-a-b)^n \to 0$ as $n \to \infty$ $\Rightarrow$ $P^n \to \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$

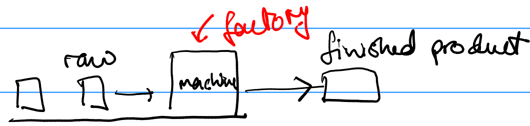
So $P^n \approx \begin{bmatrix} \pi \\ \pi \end{bmatrix}$ stationary prob. vector.

where π is the eigenvector corresponding to $\lambda = 1$

This is called the stationary probability vector.

This is a general phenomenon (of P^n converging to a set of row vectors in the limit)

Example of a system this applies to:



Two states for finished product: $\{\text{defective}, \text{good}\} =: S$

good $=1$, defective $=0$

X_n = state of n^{th} product.

Something goes wrong at some point.

Then its likely that X_{n+1} is also defective

$P(X_{n+1}=1 \mid X_n=1) = 0.99$ if product is good it remains good whp.

$P(X_{n+1}=0 \mid X_n=0) = 0.88$

There is a chance that QC finds that something is amiss and fixes it $P(X_{n+1}=1 \mid X_n=0) = 1-0.88$

$$P = \begin{bmatrix} 0.88 & 0.12 \\ 0.01 & 0.99 \end{bmatrix} = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix}$$

In the long run,

$$P^n \approx \begin{array}{cc} \text{being defective} & \text{being good} \\ \left[\begin{matrix} \frac{0.01}{0.01+0.12} & \frac{0.12}{0.13} \\ \frac{0.01}{0.13} & \frac{0.12}{0.13} \end{matrix}\right] & \end{array}$$

So in the long run, whatever state you start in you will be in "good" with probability $\frac{12}{13}$ and "defective" with prob $\frac{1}{13}$.

3.6 More examples of MCs.

Independent rv Markov chain : for HW and]

Successive maxima (☆ good HW)

One-d rvs and Gamble's ruin.]

Success runs

Suppose $\{X_n\}_{n=0}^{\infty}$ are iid such that

$X_n \in \{1, \dots, k\}$ with $\mathbb{P}(X_n = i) = p(i)$

Then

$$\mathbb{P}(X_n = i \mid X_{n-1} = s) = p(i) \quad \forall\, i$$

$$\Rightarrow \quad P = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & k \end{array} \\ \begin{bmatrix} p(1) & p(2) & \cdots & p(k) \\ - & - & - & - \\ & & & \\ p(1) & p(2) & \cdots & p(k) \end{bmatrix} \end{array}$$

all the rows are the same.

What should happen to P^n ?

Well, $P^n_{ij} = P(X_n = j \mid X_0 = i) = P(X_n = j)$

independence.

$$= p(j)$$

Again its independent of i the row index, and hence again $\lim_{n \to \infty} P^n = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}$

where π is a row vector. $\pi = (p(1), p(2), \dots p(k))$

Gamblers ruin (redux)

$S = \{0, \ldots, N\}$ $\quad \underbrace{p(0,0) = 1 \quad p(N,N) = 1}_{\text{absorbing states.}}$

$p(i,i+1) = p \quad p(i,i-1) = q \quad , \quad p+q = 1$

$T = \min\{n \geq 0 \mid X_n = 0 \text{ or } N\}$ time of "ruin"

Set $u_k = P(X_T = 0 \mid X_0 = k)$ prob. that I am ruined if I start with wealth k.

Then the usual 1st step analysis says

☆ $u_k = p\,u_{k+1} + q\,u_{k-1}$ when $1 \leq k \leq N-1$

$u_0 = 1$ and $u_N = 0$ ← boundary conditions.

1 $p+q = 1 \Rightarrow \underbrace{(p+q)u_k}_{\text{LHS}} = \underbrace{p\,u_{k+1} + q\,u_{k-1}}_{\text{RHS}}$

2 $\Rightarrow q(u_k - u_{k-1}) = p(u_{k+1} - u_k)$ $\quad k = 1, \ldots, N-1$

☆1 $q(u_1 - u_0) = p(u_2 - u_1)$ and $q(u_{N-1} - u_{N-2}) = p(u_N - u_{N-1})$

Set $x_k = u_k - u_{k-1}$ $\quad k = 1, \ldots, N$

☆1 → $x_k = \frac{p}{q}\overbrace{x_{k+1}}^{= p/q\, x_{k+2}}$ $\quad k = 1, \ldots, N-1$

$= \left(\frac{p}{q}\right)^{N-k} x_N$

$x_{N-1} = \left(p/q\right) x_N$

$x_{N-2} = \left(p/q\right)^2 x_N$

Set $k=1$ to get $x_1 = u_1 - u_0 = u_1 - 1$

$$\Rightarrow u_1 = 1 + \left(\frac{p}{q}\right)^{N-1} x_N$$

$$x_2 = u_2 - u_1 = \left(\frac{p}{q}\right)^{N-2} x_N \quad \Bigg| \; u_2 = \left[\left(\frac{p}{q}\right)^{N-1} + \left(\frac{p}{q}\right)^{N-2}\right] x_N + 1$$

$$u_{N-1} = \left[\left(\frac{p}{q}\right)^{N-1} + \cdots \left(\frac{p}{q}\right)^{1} + 1\right] x_N$$

*3

$$x_N = u_N - u_{N-1} = 0 - u_{N-1} = -\left[\left(\frac{p}{q}\right)^{N-1} + \cdots \left(\frac{p}{q}\right)^{1}\right] x_N - 1$$

*4

$$\Rightarrow x_N = \frac{-1}{1 + p/q + \cdots (p/q)^{N-1}}$$

*6

$$\Rightarrow u_k = 1 - \frac{\left((p/q)^{N-1} + \cdots + (p/q)^{N-k}\right)}{p/q + \cdots + (p/q)^{N-1}}$$

When $p/q = 1$, we get

$$u_k = 1 - \frac{k}{N} = \frac{N-k}{N}$$ (which we computed if we did Martingales)

*7 $p \neq q$

Otherwise

$$u_k = 1 - \frac{(p/q)^{N-k}\left((p/q)^{k} - 1\right)}{(p/q)^{N} - 1}$$

$$= \frac{(p/q)^{N-k} - 1}{(p/q)^{N} - 1}$$

Similarly, you can also find $E[T]$ for this problem using "1st step analysis"

3.6.1 treats the "general" random walk, where $p+q<1$. This case can be solved as well (and its not as interesting, so we will skip it)

Note the following: If $N = +\infty$ (the casino has $+\infty$ money) then

$$\text{※8} \quad \left[u_k = \begin{cases} 1 & \text{when } p \le q \quad \text{(Ruin is certain)} \\ \left(q/p\right)^k & p > q \quad \text{(game is favorable to player)} \end{cases} \right.$$

$q = 0.49$, $p = 0.51$, $k = 100$, $u_k = 0.018$ (Prob. of ruin is really low)

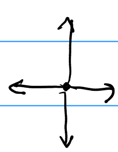
There is a +ve probability that your fortune increases without end if $p > q$.

A random walk is a discrete version of Brownian motion.

particle

random collisions (equally likely to push particle in either direction)

In general dimensions d,



$2d$ directions.

$$\mathbb{P}(X_n - X_{n-1} = e_i) = p_i \qquad i = 1, \dots, 2d$$

Success Runs Markov Chain

Flip a coin H T H H H T H H H H

My Markov chain X_n is going to count the current length of the "run" of consecutive heads

Ex: × H T H H H T H H H H

$$\begin{array}{ccccccccccc} X_0 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ \shortparallel & & & & & & & & & & \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \end{array}$$

At any stage you have probability $\frac{1}{2}$ of going from state k to $k+1$ and $\frac{1}{2}$ for going from k to 0.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \cdots \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

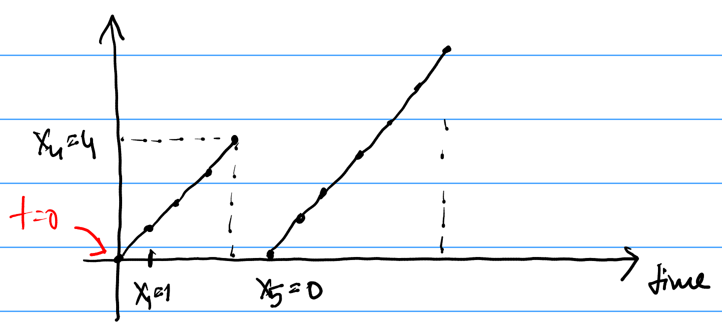
In general

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \end{array} \\ \begin{bmatrix} p_0 & q_0 & 0 & \cdots \\ p_1 & r_1 & q_1 & \\ p_2 & 0 & r_2 & q_2 \\ \vdots & & & \end{bmatrix} \end{array}$$

Ex: Suppose you have a light bulb that has lifetime ξ. $\mathbb{P}(\xi = k) = a_k$ $k = 0, 1, 2, \ldots$ (discrete). If a bulb expires then its replaced by an identical light bulb with independent lifetime

Let X_n = "current age of light bulb"

Then X_n is a success run Markov chain.



This is an example of a RENEWAL PROCESS.

☆ HW problem (based on 3.6.3) Consider the success run Markov chain. Find an equation for $E[T_5]$ the expected first time at which you see 5 consecutive heads.

<u>More HW</u>

3.6.2 Exercise, 3.6.6 (Challenging will need a hint)

3.7 Another approach to 1st step analysis using

$$(I-Q)^{-1} = I + Q + Q^2 + \cdots$$

They rederive equations like

$$(I-Q)U_k = P_k \qquad (I-Q)V = \vec{1} \qquad \text{and so on}$$

There we had argued the "renewal" property without fully justifying it rigorously:

$$E[T \mid X_0 = i, X_1 = j] = 1 + E[T \mid X_0 = j]$$

☆ Worth constructing a HW problem that shows this?